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Adaptive Attitude Control of Spacecraft: Mixed H_2/H_∞ Approach

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An adaptive mixed H_2/H_∞ attitude control of nonlinear spacecraft systems with unknown or uncertain inertia matrix and external disturbances is presented. The design objective is to specify a controller with a parameter adaptive law such that the adaptive H_2 optimal control performance can be achieved under a desired adaptive H_∞ disturbance attenuation constraint. It can be derived that the spacecraft model satisfies the properties of linear parameterization and skew symmetry. An explicit solution to the adaptive mixed H_2/H_∞ attitude control problem can be obtained by combining nonlinear minimax theory and linear quadratic optimal control techniques. Moreover, by virtue of the skew symmetric property of the spacecraft system and adequate choice of state variable transformation, this adaptive control problem can be reduced to solving two algebraic equations instead of a pair of coupled time-varying differential equations. Furthermore, with some simplification, a closed-form solution to these two algebraic equations can lead to a very simple adaptive controller that can be viewed as a mixed H_2/H_∞ proportional and derivative controller with adaptive law. Finally, experimental simulation results based on the Republic of China Satellite-1 spacecraft system are presented to demonstrate the effectiveness of the proposed design methods.

Nomenclature

(b_1, b_2, b_3)	=	body fixed reference frame
(e_1, e_2, e_3)	=	inertial fixed reference coordinates
h	=	total spacecraft angular momentum in body axes, $[h_1 \ h_2 \ h_3]^T$, $J\omega$
J	=	inertia matrix
(o_1, o_2, o_3)	=	orbital coordinate reference
P	=	weighting matrix of attitude state with respect to the initial time
Q	=	weighting matrix of attitude state
Q_f	=	weighting matrix of attitude state with respect to the final time
Q_1	=	weighting matrix of attitude state for the H_∞ performance
Q_{1f}	=	weighting matrix of attitude state with respect to the final time for the H_∞ performance
Q_2	=	weighting matrix of attitude state for the H_2 performance
Q_{2f}	=	weighting matrix of attitude state with respect to the final time for the H_2 performance
r	=	filtered link of attitude state
S_{ai}	=	saturation value of the ith actuator output torque
S_1	=	weighting matrix of parameter estimation error for the H_∞ performance
S_2	=	weighting matrix of parameter estimation error for the H_2 performance
T	=	state-space transformation matrix
$\boldsymbol{u}$	=	control torque vector
$\boldsymbol{u}_e$	=	mixed H_2/H_∞ control torque vector
W	=	weighting matrix of control torque
W_1	=	weighting matrix of control torque for the H_∞ performance
W_2	=	weighting matrix of control torque for the H_2 performance
$\boldsymbol{x}$	=	attitude state vector
γ	=	desired disturbance attenuation level
θ	=	attitude Euler angle, $[\theta_1 \ \theta_2 \ \theta_3]^T$
$[\nu\times]$	=	cross product matrix associated with the vector $\boldsymbol{\nu} = [\nu_1 \quad \nu_2 \quad \nu_3]^T$, $\begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix}$
$\boldsymbol{\tau}_a$	=	torque vector due to actuator, $[\tau_{a1} \ \tau_{a2} \ \tau_{a3}]^T$
$\boldsymbol{\tau}_{\text{aero}}$	=	aerodynamic disturbance torque vector
$\boldsymbol{\tau}_d$	=	external disturbance torque vector
$\boldsymbol{\tau}_g$	=	gravity gradient torque vector
$\boldsymbol{\tau}_{\text{solar}}$	=	solar radiation pressure disturbance torque vector
Ω	=	bounded region, $\{\boldsymbol{x} \in R^3 \mid -\pi/2 < \theta_2 < \pi/2\}$
$\boldsymbol{\omega}$	=	general angular velocity vector in body axes, $[\omega_1 \ \omega_2 \ \omega_3]^T$
ω_0	=	orbital rate

Introduction

THE attitude control of spacecraft has received extensive attention in recent decades, and several methods of the spacecraft attitude control have been developed to treat this problem. Based on linearization using coordinate transformation and nonlinear feedback, a controller for attitude control has been derived.[1] The feasibility of applying the feedback linearization technique to spacecraft attitude control and the momentum management problem has also been discussed.[2,3] A sliding manifold approach[4,5] has also been used for spacecraft attitude control. More relevant to this study, optimal control theory has been used for attitude control of spacecraft systems.[6–8] The approach of H_∞ control has been applied to the space station attitude and momentum control problem while taking into consideration the linearized equations of motion.[9] Recently, Chen and Wu[10] developed a nonlinear H_∞ control design to treat the spacecraft attitude control problem under parameter perturbation and external noise.

Over the past 10 years, mixed H_2/H_∞ optimal control has been studied for linear systems.[11–15] The main purpose of this type of control is to design an H_2 optimal control for the worst-case external disturbance whose effects on system output must be attenuated below a desired value, that is, to design an H_2 optimal control under the H_∞ disturbance attenuation constraint. This control design is very suitable for the optimal control problem of systems under uncertain external disturbance. Along this line, based on the nonlinear two-player Nash differential game theorem, Wu et al.[16] also developed a unified design for H_2, H_∞, and mixed H_2/H_∞ attitude control of spacecraft.

However, in a practical situation, system parameter variations such as those generated by payload motion, vehicle docking, the

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rotation of solar arrays, and so on, are inevitable. One can develop an adaptive scheme that will continuously adjust controller gains to compensate for parameter uncertainties and environmental effects to ensure stable and robust performance. For this problem, adaptive attitude control has been presented for spacecraft systems.[17–22]

In this paper, we design an attitude controller with a parameter adaptive law so that the adaptive H_2 optimal control performance can be achieved under a desired adaptive H_∞ disturbance attenuation constraint when the inertia matrix of spacecraft is uncertain (or unknown) and an external disturbance exists. A pair of coupled time-varying differential equations with an adaptive constraint must first be solved to design the controller. By an adequate choice of a state transformation and use of the skew-symmetric property of spacecraft systems, two time-varying coupled differential equations can be transformed into two corresponding algebraic coupled equations, and the adaptive constraint holds. Then, through Cholesky factorization (see Ref. 23), these two coupled algebraic equations can be solved easily. As a result, the structure of the controller becomes very simple and can be viewed as a mixed H_2/H_∞ adaptive proportional and derivative (PD) controller with the control gains depending on the disturbance attenuation level γ, which is assigned according to mission requirements.

Mathematical Model and Problem Formulation

Spacecraft Model

Consider a spacecraft moving in a circular orbit. The coordinate systems used in the attitude control are shown in Fig. 1. The inertial fixed reference coordinates, e_1, e_2, and e_3, with their origin at the center of the Earth are used to determine the orbital position of the spacecraft. The orbital coordinate reference, o_1, o_2, and o_3, is rotating about the o_2 axis with respect to the fixed inertial coordinate system e_1, e_2, and e_3 at the orbital rate ω_0. The axes of this reference frame are chosen such that the roll axis o_1 is in the flight direction, the pitch axis o_2 is perpendicular to the orbital plane, and the yaw axis o_3 points toward the Earth. The last reference system used is the body-fixed reference frame, b_1, b_2, and b_3. The orientation of the spacecraft with respect to the reference frame, o_1, o_2, and o_3, is obtained, in this work, by a yaw-pitch-roll (θ_3-θ_2-θ_1) sequence of rotations. The origins of both orbit coordinates and body-fixed coordinates are at the center of mass of the spacecraft.

The nonlinear equations of motion, in terms of components along the body-fixed control axes, are given by the attitude kinematics and the spacecraft dynamics and can be written as[24,25]

$$\omega = R(\theta)\dot{\theta} - \omega_c(\theta) \tag{1}$$

for attitude kinematics and

$$J\dot{\omega} = [h\times]\omega + \tau_g + \tau_a + \tau_d \tag{2}$$

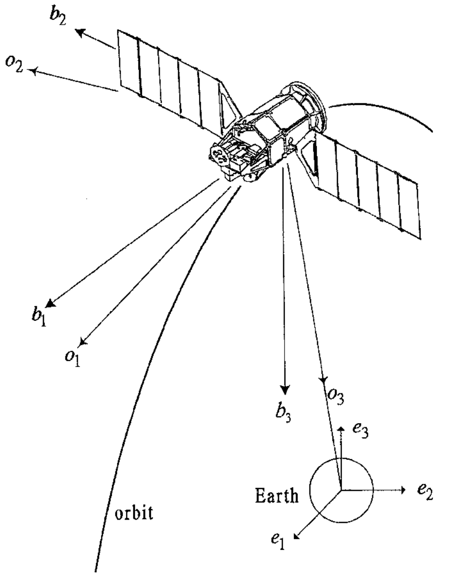

Fig. 1 ROCSAT-1 spacecraft and coordinate systems used in the spacecraft attitude control.

for spacecraft dynamics where

$$R(\theta) = \begin{bmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2 \\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$\omega_c(\theta) = \omega_0 \begin{bmatrix} \cos\theta_2\sin\theta_3 \\ \cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_3 + \cos\theta_1\sin\theta_2\sin\theta_3 \end{bmatrix}$$

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \qquad \boldsymbol{\tau}_g = 3\omega_0^2[c\times]Jc$$

with

$$c = \begin{bmatrix} -\sin\theta_2 \\ \sin\theta_1\cos\theta_2 \\ \cos\theta_1\cos\theta_2 \end{bmatrix}$$

and $\boldsymbol{\tau}_d = \boldsymbol{\tau}_{\text{aero}} + \boldsymbol{\tau}_{\text{solar}}$.

Remark 1: If the control torque is limited by the saturation of actuator, the spacecraft dynamics (2) is of the following form:

$$J\dot{\omega} = [h\times]\boldsymbol{\omega} + \boldsymbol{\tau}_g + \boldsymbol{\tau}_{a,\text{sat}} + \boldsymbol{\tau}_{\text{aero}} + \boldsymbol{\tau}_{\text{solar}} \tag{3}$$

where $\boldsymbol{\tau}_{a,\text{sat}} = [\text{sat}(\boldsymbol{\tau}_{a1})\ \text{sat}(\boldsymbol{\tau}_{a2})\ \text{sat}(\boldsymbol{\tau}_{a3})]^T$
and

$$\text{sat}(\boldsymbol{\tau}_{ai}) = \begin{cases} S_{ai}, \boldsymbol{\tau}_{ai} \geq S_{ai} \\ \boldsymbol{\tau}_{ai}, |\boldsymbol{\tau}_{ai}| < S_{ai} \\ -S_{ai}, \boldsymbol{\tau}_{ai} \leq -S_{ai} \end{cases} \qquad \text{for} \qquad i = 1, 2, 3$$

To apply the proposed design method, the dynamic equation (3) can be put into the form of Eq. (2) with $\boldsymbol{\tau}_d = \boldsymbol{\tau}_{\text{aero}} + \boldsymbol{\tau}_{\text{solar}} + \boldsymbol{\tau}_{a,\text{sat}} - \boldsymbol{\tau}_a$. In this situation, the deviation $\boldsymbol{\tau}_{a,\text{sat}} + \boldsymbol{\tau}_a$ can be considered as a part of the external disturbance. □

Remark 2: This description of Eq. (1) is defined for all θ_1, θ_2, and θ_3 except $\theta_2 = \pm(2n+1)(\pi/2)$ for any integer n. The singularity [i.e., the determinant of matrix $R(\theta)$ becomes zero at $\theta_2 = \pm(2n+1)(\pi/2)$] arises owing to the choice of the set of rotations that define the orientation of the spacecraft relative to the orbital frame. However, when the orientation corresponding to the singularity at $\theta_2 = \pm\pi/2$ lies in the control region of attitude angles, another set of rotations can be defined to eliminate this singularity.[26] In this paper, we are interested in the attitudes in the bounded region Ω. □

Differentiating Eq. (1) gives

$$\dot{\omega} = R(\theta)\ddot{\theta} + \left[\frac{\mathrm{d}}{\mathrm{d}t}R(\theta)\right]\dot{\theta} - \left[\frac{\mathrm{d}}{\mathrm{d}t}\omega_c(\theta)\right] \tag{4}$$

Substituting ω and $\dot{\omega}$ from Eqs. (1) and (4) into Eq. (2) and premultiplying Eq. (4) by the matrix $R^T(\theta)$, we obtain

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta,\dot{\theta}) = \boldsymbol{u} + d \tag{5}$$

where

$$M(\theta) = R^T(\theta)JR(\theta)$$

$$C(\theta,\dot{\theta}) = R^T(\theta)J\left[\frac{\mathrm{d}}{\mathrm{d}t}R(\theta)\right] - R^T(\theta)[h\times]R(\theta)$$

$$G(\theta,\dot{\theta}) = -R^T(\theta)J\left[\frac{\mathrm{d}}{\mathrm{d}t}\omega_c(\theta)\right] + R^T(\theta)[h\times]\omega_c(\theta) - 3\omega_0^2R^T(\theta)[c\times]Jc$$

$$u = R^T(\theta)\boldsymbol{\tau}_a, \qquad \mathrm{d} = R^T(\theta)\boldsymbol{\tau}_d$$

In this paper, we consider the nonlinear spacecraft model in Eq. (5) under the following assumption.

Assumption: The nominal inertia matrix J is an unknown constant and is symmetric positive definite. Moreover, the external disturbance $\boldsymbol{\tau}_d$ is bounded but unknown. □

Properties: Under the preceding assumption, the spacecraft system (5) has the following properties.

1) The matrix $M(\theta)$ is symmetric positive definite.

2) The matrix $\frac{1}{2}(\mathrm{d}/\mathrm{d}t)M(\theta) - C(\theta,\dot{\theta})$ is skew symmetric,[27] that is,

$$x^T\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}M(\theta) - C(\theta,\dot{\theta})\right]\boldsymbol{x} = 0, \qquad \forall \boldsymbol{x} \in R^3 \tag{6}$$

3) The spacecraft parameter matrices M, C, and G form the following linear parameterization property

$$M(\theta)\dot{y} + C(\theta,\dot{\theta})y + G(\theta,\dot{\theta}) = \Phi(\theta,\dot{\theta},y,\dot{y})\boldsymbol{\xi} \tag{7}$$

where $\boldsymbol{\xi} = [J_{11}\ J_{12}\ J_{13}\ J_{22}\ J_{23}\ J_{33}]^T$ denotes the unknown constant parameter vector and the known regression matrix $\Phi(\theta,\dot{\theta},y,\dot{y})$ is described in the Appendix.

Problem Formulation

In general, precise knowledge about system parameters M, C, and G is required while calculating an appropriate applied torque $\boldsymbol{\tau}_a$ in the mixed H_2/H_∞ attitude control problem of the given spacecraft systems.[16] In this paper, we deal with the adaptive mixed H_2/H_∞ attitude control problem of the given spacecraft systems under unknown inertia matrix J and external disturbance $\boldsymbol{\tau}_d$. In this case, the control law must be identified and adapted to the operation conditions.

If we define the controlled state as follows

$$x = \begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} \tag{8}$$

and introduce a filtered link of state r and a state-space transformation matrix T as

$$r = \lambda_1\theta + \lambda_2\dot{\theta} \tag{9a}$$

$$T = \begin{bmatrix}\lambda_1 I_{3\times3} & \lambda_2 I_{3\times3}\\ I_{3\times3} & 0_{3\times3}\end{bmatrix} \tag{9b}$$

for some positive constants λ_1 and λ_2, which should be adequately determined later, then we have

$$\dot{r} = -M^{-1}(\theta)C(\theta,\dot{\theta})r + \lambda_2 M^{-1}(\theta)\{\Phi[\theta,\dot{\theta},(\lambda_1/\lambda_2)\theta, (\lambda_1/\lambda_2)\dot{\theta}]\boldsymbol{\xi} + \boldsymbol{u} + d\} \tag{10}$$

where

$$\Phi(\boldsymbol{x},t)\,\boldsymbol{\xi} = \Phi[\theta,\dot{\theta},(\lambda_1/\lambda_2)\theta,(\lambda_1/\lambda_2)\dot{\theta}]\,\boldsymbol{\xi}$$
$$= (\lambda_1/\lambda_2)M(\theta)\dot{\theta} + (\lambda_1/\lambda_2)C(\theta,\dot{\theta})\theta - G(\theta,\dot{\theta})$$

Thus, by Eqs. (8–10), the dynamic equation of the spacecraft attitude control system can be obtained as

$$\dot{\boldsymbol{x}}(t) = T^{-1}\begin{bmatrix}\dot{r}(t)\\ \dot{\theta}(t)\end{bmatrix} = A_T(\boldsymbol{x},t)\boldsymbol{x} + B_T(\boldsymbol{x},t)[\Phi(\boldsymbol{x},t)\,\boldsymbol{\xi} + \boldsymbol{u}] + B_T(\boldsymbol{x},t)d \tag{11}$$

where

$$A_T(\boldsymbol{x},t) = \begin{bmatrix}0_{3\times3} & I_{3\times3}\\ -(\lambda_1/\lambda_2)M^{-1}(\theta)C(\theta,\dot{\theta}) & -M^{-1}(\theta)C(\theta,\dot{\theta}) - (\lambda_1/\lambda_2)I_{3\times3}\end{bmatrix}$$

$$B_T(\boldsymbol{x},t) = \begin{bmatrix}0_{3\times3}\\ M^{-1}(\theta)\end{bmatrix}$$

If the following applied torque is selected,

$$\boldsymbol{u} = \boldsymbol{u}_e - \Phi(\boldsymbol{x},t)\,\hat{\boldsymbol{\xi}} \tag{12}$$

then the dynamic equation driven by the control torque $\boldsymbol{u}_e$ becomes

$$\dot{\boldsymbol{x}} = A_T(\boldsymbol{x},t)\boldsymbol{x} + B_T(\boldsymbol{x},t)[\Phi(\boldsymbol{x},t)\,\tilde{\boldsymbol{\xi}} + \boldsymbol{u}_e + d] \tag{13}$$

where $\tilde{\boldsymbol{\xi}} =: \boldsymbol{\xi} - \hat{\boldsymbol{\xi}}$ denotes the estimation error.

Formally, the adaptive attitude control problem that is considered in this paper for the dynamic equation of spacecraft system in Eq. (13) can be stated as follows.

Adaptive Mixed H_2/H_∞ Attitude Control Problem

Consider the nonlinear spacecraft dynamic system (13). Given a desired disturbance attenuation level $\gamma > 0$ and weighting matrices Q_1, Q_2, W_1, and W_2, the adaptive mixed H_2/H_∞ attitude control problem is said to be solved if there exists a control law $\boldsymbol{u}_e(t)$ and an adaptive law of $\hat{\boldsymbol{\xi}}(t)$ such that the following H_2 (quadratic) optimal attitude control performance[11–15]

$$\min_{\boldsymbol{u}_e(t)\in L_2[0,t_f]}\left[\boldsymbol{x}^T(t_f)Q_{2f}\,\boldsymbol{x}(t_f) + \tilde{\boldsymbol{\xi}}^T(t_f)S_2\tilde{\boldsymbol{\xi}}(t_f) + \int_0^{t_f}\left[\boldsymbol{x}^T(t)Q_2\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W_2\boldsymbol{u}_e(t)\right]\mathrm{d}t\right] \tag{14}$$

can be achieved for all $t_f \in [0,\infty)$ and for some positive-definite matrices $Q_{2f} = Q_{2f}^T > 0$ and $S_2 = S_2^T > 0$ under the following H_∞ disturbance attenuation constraint:

$$\boldsymbol{x}^T(t_f)Q_{1f}\boldsymbol{x}(t_f) + \tilde{\boldsymbol{\xi}}^T(t_f)S_1\tilde{\boldsymbol{\xi}}(t_f) + \int_0^{t_f}\left[\boldsymbol{x}^T(t)Q_1\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W_1\boldsymbol{u}_e(t)\right]\mathrm{d}t \le \boldsymbol{x}^T(0)P\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S_1\tilde{\boldsymbol{\xi}}(0) + \gamma^2\int_0^{t_f}\mathrm{d}^T(t)\,\mathrm{d}(t)\,\mathrm{d}t \qquad \forall\,\mathrm{d}(t)\in L_2[0,t_f] \tag{15}$$

for some positive-definite matrices $Q_{1f} = Q_{1f}^T > 0$, $S_1 = S_1^T > 0$, and $P = P^T > 0$. That is, not only the H_2 optimal attitude control performance is achieved, but also the effect of disturbance $\mathrm{d}(t)$ on the state $\boldsymbol{x}(t)$ and control $\boldsymbol{u}_e(t)$ must be attenuated below a desired level γ^2 from the energy point of view. □

Remark 3: Let

$$J_1(\boldsymbol{u}_e,d) = \boldsymbol{x}^T(t_f)Q_{1f}\,\boldsymbol{x}(t_f) + \tilde{\boldsymbol{\xi}}^T(t_f)S_1\tilde{\boldsymbol{\xi}}(t_f) + \int_0^{t_f}\left[\boldsymbol{x}^T(t)Q_1\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W_1\boldsymbol{u}_e(t) - \gamma^2\,\mathrm{d}^T(t)\,\mathrm{d}(t)\right]\mathrm{d}t \tag{16a}$$

$$J_2(\boldsymbol{u}_e,d) = \boldsymbol{x}^T(t_f)Q_{2f}\,\boldsymbol{x}(t_f) + \tilde{\boldsymbol{\xi}}^T(t_f)S_2\tilde{\boldsymbol{\xi}}(t_f) + \int_0^{t_f}\left[\boldsymbol{x}^T(t)Q_2\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W_2\boldsymbol{u}_e(t)\right]\mathrm{d}t \tag{16b}$$

Then, the adaptive mixed H_2/H_∞ attitude control with both performances (14) and (15) is equivalent to find the control law $\boldsymbol{u}_e^*(\boldsymbol{x},t)$, the adaptive law of $\hat{\boldsymbol{\xi}}(\boldsymbol{x},t)$, and the worst-case disturbance $d^*(\boldsymbol{x},t)$ such that[28]

$$J_1\left[\boldsymbol{u}_e^*(\boldsymbol{x},t), d^*(\boldsymbol{x},t)\right] \ge J_1\left[\boldsymbol{u}_e^*(\boldsymbol{x},t), d\right] \qquad \forall\, d\in L_2[0,t_f] \tag{17a}$$

$$J_2\left[\boldsymbol{u}_e^*(\boldsymbol{x},t), d^*(\boldsymbol{x},t)\right] \le J_2\left[\boldsymbol{u}_e, d^*(\boldsymbol{x},t)\right] \qquad \forall\, \boldsymbol{u}_e\in L_2[0,t_f] \tag{17b}$$

This mixed performance with parameter estimation can be viewed as an adaptive attitude control design approach that minimizes an H_2 cost function under the H_∞ disturbance attenuation constraint on the spacecraft attitude controlled system associated with the unknown parameter and external disturbance. □

By use of a state transformation and the property of symmetry, the given adaptive mixed H_2/H_∞ attitude control problem is solved as follows.

Adaptive Mixed H_2/H_∞ Attitude Control Design

Sufficient Conditions of the Adaptive Mixed H_2/H_∞ Attitude Control Problem

In this section, we present sufficient conditions for the existence of the solution of an adaptive mixed H_2/H_∞ attitude control problem. For the convenience of design, we take $W_1 = W_2 = W$ and $S_1 = S_2 = S$ throughout this study. By using the standard technique of completing the squares, we have the following theorem.

Theorem: For the dynamic equation of spacecraft attitude control system in Eq. (11), if the adaptive mixed H_2/H_∞ control law $\boldsymbol{u}^*(\boldsymbol{x}, t)$ is chosen as

$$\boldsymbol{u}^*(\boldsymbol{x}, t) = \boldsymbol{u}_e^*(\boldsymbol{x}, t) - \Phi(\boldsymbol{x}, t)\,\hat{\boldsymbol{\xi}}(\boldsymbol{x}, t) \tag{18a}$$

where

$$\boldsymbol{u}_e^*(\boldsymbol{x}, t) = -W^{-1}B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x} \tag{18b}$$

$$\dot{\hat{\boldsymbol{\xi}}}(\boldsymbol{x}, t) = S^{-1}\Phi^T(\boldsymbol{x}, t)B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x} \tag{18c}$$

and $P_1(\boldsymbol{x}, t)$ and $P_2(\boldsymbol{x}, t)$ are the solutions of the following coupled time-varying differential equations

$$\begin{aligned}&\dot{P}_1(\boldsymbol{x}, t) + P_1(\boldsymbol{x}, t)A_T(\boldsymbol{x}, t) + A_T(\boldsymbol{x}, t)^T P_1(\boldsymbol{x}, t) + Q_1\\ &\quad - [P_1(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t) \quad P_2(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)]\\ &\quad \times \begin{bmatrix}(-1/\gamma^2)I & W^{-1}\\ W^{-1} & -W^{-1}\end{bmatrix}\begin{bmatrix}B_T(\boldsymbol{x}, t)^T P_1(\boldsymbol{x}, t)\\ B_T(\boldsymbol{x}, t)^T P_2(\boldsymbol{x}, t)\end{bmatrix} = 0\end{aligned} \tag{19a}$$

$$\begin{aligned}&\dot{P}_2(\boldsymbol{x}, t) + P_2(\boldsymbol{x}, t)A_T(\boldsymbol{x}, t) + A_T(\boldsymbol{x}, t)^T P_2(\boldsymbol{x}, t) + Q_2\\ &\quad - [P_1(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t) \quad P_2(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)]\\ &\quad \times \begin{bmatrix}0_{3\times 3} & (-1/\gamma^2)I_{3\times 3}\\ (-1/\gamma^2)I_{3\times 3} & W^{-1}\end{bmatrix}\begin{bmatrix}B_T(\boldsymbol{x}, t)^T P_1(\boldsymbol{x}, t)\\ B_T(\boldsymbol{x}, t)^T P_2(\boldsymbol{x}, t)\end{bmatrix} = 0\end{aligned} \tag{19b}$$

and the adaptive constraint

$$B_T^T(\boldsymbol{x}, t)P_1(\boldsymbol{x}, t) = B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t) \tag{19c}$$

with $P_1(\boldsymbol{x}, t) = P_1^T(\boldsymbol{x}, t) \geq 0$, $P_2(\boldsymbol{x}, t) = P_2^T(\boldsymbol{x}, t) \geq 0$, and the terminal conditions $P_1[\boldsymbol{x}(t_f), t_f] = Q_{1f}$ and $P_2[\boldsymbol{x}(t_f), t_f] = Q_{2f}$, then the adaptive mixed H_2/H_∞ attitude control problem (17a) under (17b) is solved by Eqs. (18a–18d). Also, the worst-case disturbance $d^*(\boldsymbol{x}, t)$ is

$$d^*(\boldsymbol{x}, t) = (1/\gamma^2)B_T^T(\boldsymbol{x}, t)P_1(\boldsymbol{x}, t)\,\boldsymbol{x} \tag{20}$$

□

Proof: Let us first consider the cost function $J_2(\boldsymbol{u}_e, d)$ in Eq. (16b). Obviously, it can be rewritten as

$$\begin{aligned}J_2(\boldsymbol{u}_e, d) &= \boldsymbol{x}^T(t_f)Q_{2f}\,\boldsymbol{x}(t_f) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0) + \int_0^{t_f}\Big[\boldsymbol{x}^T(t)Q_2\,\boldsymbol{x}(t)\\ &\quad + \boldsymbol{u}_e^T(t)W\boldsymbol{u}_e(t) + \frac{\mathrm{d}}{\mathrm{d}t}\big[\boldsymbol{x}^T(t)P_2(\boldsymbol{x}, t)\boldsymbol{x}(t) + \tilde{\boldsymbol{\xi}}^T(t)S\tilde{\boldsymbol{\xi}}(t)\big]\Big]\,\mathrm{d}t\\ &\quad - \boldsymbol{x}^T(t_f)P_2[\boldsymbol{x}(t_f), t_f]\,\boldsymbol{x}(t_f) + \boldsymbol{x}^T(0)P_2[\boldsymbol{x}(0), 0]\,\boldsymbol{x}(0)\end{aligned}$$

By the terminal condition $P_2[\boldsymbol{x}(t_f), t_f] = Q_{2f}$, we obtain

$$\begin{aligned}J_2(\boldsymbol{u}_e, d) &= \boldsymbol{x}^T(0)P_2[\boldsymbol{x}(0), 0]\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0)\\ &\quad + \int_0^{t_f}\big[\boldsymbol{x}^T(t)Q_2\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W\boldsymbol{u}_e(t) + \dot{\boldsymbol{x}}^T(t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\\ &\quad + \boldsymbol{x}^T(t)\dot{P}_2(\boldsymbol{x}, t)\boldsymbol{x}(t) + \boldsymbol{x}^T(t)P_2(\boldsymbol{x}, t)\,\dot{\boldsymbol{x}}(t)\\ &\quad + \dot{\tilde{\boldsymbol{\xi}}}^T(t)S\tilde{\boldsymbol{\xi}}(t) + \tilde{\boldsymbol{\xi}}^T(t)S\dot{\tilde{\boldsymbol{\xi}}}(t)\big]\,\mathrm{d}t\end{aligned} \tag{21}$$

Substituting the dynamic equation (13) into Eq. (21) leads to

$$\begin{aligned}J_2(\boldsymbol{u}_e, d) &= \boldsymbol{x}^T(0)P_2[\boldsymbol{x}(0), 0]\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0)\\ &\quad + \int_0^{t_f}\big\{\boldsymbol{x}^T(t)\big[\dot{P}_2(\boldsymbol{x}, t) + P_2(\boldsymbol{x}, t)A_T(\boldsymbol{x}, t) + A_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\\ &\quad + Q_2\big]\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W\boldsymbol{u}_e(t) + \boldsymbol{u}_e^T(t)B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\\ &\quad + \boldsymbol{x}^T(t)P_2(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)\,\boldsymbol{u}_e(t) + \mathrm{d}^T(t)B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\\ &\quad + \boldsymbol{x}^T(t)P_2(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)\,\mathrm{d}(t) + \dot{\tilde{\boldsymbol{\xi}}}^T(t)S\tilde{\boldsymbol{\xi}}(t) + \tilde{\boldsymbol{\xi}}^T(t)S\dot{\tilde{\boldsymbol{\xi}}}(t)\\ &\quad + \tilde{\boldsymbol{\xi}}^T(t)\Phi^T(\boldsymbol{x}, t)B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\\ &\quad + \boldsymbol{x}^T(t)P_2(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)\Phi(\boldsymbol{x}, t)\tilde{\boldsymbol{\xi}}(t)\big\}\,\mathrm{d}t\end{aligned} \tag{22}$$

Thus, by the worst-case disturbance $d^*(\boldsymbol{x}, t)$ in Eq. (20), the adaptive law $\hat{\boldsymbol{\xi}}(\boldsymbol{x}, t)$ in Eq. (18c) given that $\dot{\tilde{\boldsymbol{\xi}}}(\boldsymbol{x}, t) = -\dot{\hat{\boldsymbol{\xi}}}(\boldsymbol{x}, t)$ and the differential equation (19b), we have

$$\begin{aligned}J_2\big[\boldsymbol{u}_e, d^*(\boldsymbol{x}, t)\big] &= \boldsymbol{x}^T(0)P_2[\boldsymbol{x}(0), 0]\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0)\\ &\quad + \int_0^{t_f}\big\{\big[\boldsymbol{u}_e(t) + W^{-1}B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\big]^T\\ &\quad \times W\big[\boldsymbol{u}_e(t) + W^{-1}B_T^T(\boldsymbol{x}, t)P_2(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\big]\big\}\,\mathrm{d}t\end{aligned} \tag{23}$$

which, by the control law in Eq. (18b), results in

$$J_2\big[\boldsymbol{u}_e^*(\boldsymbol{x}, t), d^*(\boldsymbol{x}, t)\big] = \boldsymbol{x}^T(0)P_2[\boldsymbol{x}(0), 0]\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0) \tag{24}$$

Then, we have

$$J_2\big[\boldsymbol{u}_e^*(\boldsymbol{x}, t), d^*(\boldsymbol{x}, t)\big] \leq J_2\big[\boldsymbol{u}_e, d^*(\boldsymbol{x}, t)\big] \qquad \forall\,\boldsymbol{u}_e(t) \in L_2[0, t_f] \tag{25}$$

Similarly, the cost function $J_1(\boldsymbol{u}_e, d)$ in Eq. (16a) can be rewritten as

$$\begin{aligned}J_1(\boldsymbol{u}_e, d) &= \boldsymbol{x}^T(0)P_1[\boldsymbol{x}(0), 0]\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0)\\ &\quad + \int_0^{t_f}\big\{\boldsymbol{x}^T(t)\big[\dot{P}_1(\boldsymbol{x}, t) + P_1(\boldsymbol{x}, t)A_T(\boldsymbol{x}, t) + A_T^T(\boldsymbol{x}, t)P_1(\boldsymbol{x}, t)\\ &\quad + Q_1\big]\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W\boldsymbol{u}_e(t) + \boldsymbol{u}_e^T(t)B_T^T(\boldsymbol{x}, t)P_1(\boldsymbol{x}, t)\boldsymbol{x}(t)\\ &\quad + \boldsymbol{x}^T(t)P_1(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)\,\boldsymbol{u}_e(t) - \gamma^2\,\mathrm{d}^T(t)\,\mathrm{d}(t)\\ &\quad + \mathrm{d}^T(t)B_T^T(\boldsymbol{x}, t)P_1(\boldsymbol{x}, t)\boldsymbol{x}(t) + \boldsymbol{x}^T(t)P_1(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)\,\mathrm{d}(t)\\ &\quad + \dot{\tilde{\boldsymbol{\xi}}}^T(t)S\tilde{\boldsymbol{\xi}}(t) + \tilde{\boldsymbol{\xi}}^T(t)S\dot{\tilde{\boldsymbol{\xi}}}(t) + \tilde{\boldsymbol{\xi}}^T(t)\Phi^T(\boldsymbol{x}, t)B_T^T(\boldsymbol{x}, t)\\ &\quad \times P_1(\boldsymbol{x}, t)\boldsymbol{x}(t) + \boldsymbol{x}^T(t)P_1(\boldsymbol{x}, t)B_T(\boldsymbol{x}, t)\Phi(\boldsymbol{x}, t)\tilde{\boldsymbol{\xi}}(t)\big\}\,\mathrm{d}t\end{aligned} \tag{26}$$

It can be deduced from the differential equation (19a), the optimal control $\boldsymbol{u}_e^*(\boldsymbol{x}, t)$ in Eq. (18b), the adaptive law $\hat{\boldsymbol{\xi}}(\boldsymbol{x}, t)$ in Eq. (18c), and the relation in Eq. (19c) that

$$\begin{aligned}J_1\big[\boldsymbol{u}_e^*(\boldsymbol{x}, t), d\big] &= \boldsymbol{x}^T(0)P_1[(\boldsymbol{x}(0), 0]\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0)\\ &\quad - \int_0^{t_f}\left\{\left[\gamma\,\mathrm{d}(t) - \frac{1}{\gamma}B_T^T(\boldsymbol{x}, t)P_1(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\right]^T\right.\\ &\quad \left.\times\left[\gamma\,\mathrm{d}(t) - \frac{1}{\gamma}B_T^T(\boldsymbol{x}, t)P_1(\boldsymbol{x}, t)\,\boldsymbol{x}(t)\right]\right\}\mathrm{d}t\end{aligned} \tag{27}$$

Then, we can conclude that

$$J_1\big[\boldsymbol{u}_e^*(\boldsymbol{x}, t), d^*(\boldsymbol{x}, t)\big] = \boldsymbol{x}^T(0)P_1[\boldsymbol{x}(0), 0]\boldsymbol{x}(0) + \tilde{\boldsymbol{\xi}}^T(0)S\tilde{\boldsymbol{\xi}}(0) \tag{28}$$

$$J_1\big[\boldsymbol{u}_e^*(\boldsymbol{x}, t), d^*(\boldsymbol{x}, t)\big] \geq J_1\big[\boldsymbol{u}_e^*(\boldsymbol{x}, t), d\big], \qquad \forall\,d(t) \in L_2[0, t_f] \tag{29}$$

QED

Remark 4: Part 1 of Remark 4 is as follows. If only the adaptive H_2 optimal attitude control design is considered, the desired disturbance attenuation constraint is negligible, that is, $\gamma = \infty$. In this situation, the adaptive H_2 control law is given as

$$u^*(x,t) = u_e^*(x,t) - \Phi(x,t)\hat{\xi}(x,t) \tag{30a}$$

where

$$u_e^*(x,t) = -W^{-1}B_T^T(x,t)P(x,t)x(t) \tag{30b}$$

$$\dot{\hat{\xi}}(x,t) = S^{-1}\Phi^T(x,t)B_T^T(x,t)P(x,t)x(t) \tag{30c}$$

and $P(x,t) = P^T(x,t) \geq 0$ is the solution of the following time-varying differential equation:

$$\dot{P}(x,t) + P(x,t)A_T(x,t) + A_T(x,t)^T P(x,t) + Q - P(x,t)B_T(x,t)W^{-1}B_T^T(x,t)P(x,t) = 0 \tag{31a}$$

$$P[x(t_f), t_f] = Q_f \tag{31b}$$

This result is the same as the time-varying differential equation in Eq. (19a) or Eq. (19b) with

$$P(x,t) = P_1(x,t) = P_2(x,t) \qquad Q = Q_1 = Q_2$$

$$Q_f = Q_{1f} = Q_{2f}$$

Part 2 of Remark 4 is as follows. If only the adaptive H_∞ attitude control design is considered, the criteria (17a) and (17b) are combined into the following dynamic game problem[29,30]:

$$J\left[u_e^*(x,t), d\right] \leq J\left[u_e^*(x,t), d^*(x,t)\right] \leq J\left[u_e, d^*(x,t)\right] \tag{32a}$$

with

$$J(u_e, d) = x^T(t_f)Q_f x(t_f) + \tilde{\xi}^T(t_f)S\tilde{\xi}(t_f) + \int_0^{t_f}\left[x^T(t)Qx(t) + u_e^T(t)Wu_e(t) - \gamma^2 d^T(t)d(t)\right]dt \tag{32b}$$

In this situation, the solution of the adaptive H_∞ attitude control of the spacecraft system is given as

$$u^*(x,t) = u_e^*(x,t) - \Phi(x,t)\hat{\xi}(x,t) \tag{33a}$$

where

$$u_e^*(x,t) = -W^{-1}B_T^T(x,t)P(x,t)x(t) \tag{33b}$$

$$\dot{\hat{\xi}}(x,t) = S^{-1}\Phi^T(x,t)B_T^T(x,t)P(x,t)x(t) \tag{33c}$$

and $P(x,t) = P^T(x,t) \geq 0$ is the solution of the following time-varying differential equation

$$\dot{P}(x,t) + P(x,t)A_T(x,t) + A_T(x,t)^T P(x,t) + Q - P(x,t)B_T(x,t)[W^{-1} - (1/\gamma^2)I]B_T^T(x,t)P(x,t) = 0 \tag{34a}$$

$$P[x(t_f), t_f] = Q_f \tag{34b}$$

This result is the same as the time-varying differential equation in Eq. (19a) with $P(x,t) = P_1(x,t) = P_2(x,t)$, $Q = Q_1$, and $Q_f = Q_{1f}$. □

From the preceding analysis, the main work in the design of adaptive mixed H_2/H_∞ attitude control of the spacecraft system is reduced to solve the time-varying coupled differential equations in Eqs. (19a–19c). Furthermore, because the coefficient matrix $B_T(x,t)$ in Eq. (11) is unknown, the control law and the adaptive law in Eqs. (18b) and (18c) can not be employed directly and need to be derived further.

Solution of the Time-Varying Differential Equations

In general, however, it is difficult to solve $P_1(x,t)$ and $P_2(x,t)$ in the coupled time-varying differential equations (19a) and (19b), especially satisfying the adaptive constraint in Eq. (19c) simultaneously. Fortunately, in the spacecraft system, the differential equations (19a) and (19b) can be further simplified to algebraic matrix equations by adequately selecting the nonlinear function matrices $P_1(x,t)$ and $P_2(x,t)$ and by using the skew-symmetric property in Eq. (6). Moreover, with the selected matrices $P_1(x,t)$ and $P_2(x,t)$, condition (19c) can be satisfied.

Because state transformation (9b) has been involved in the process of design, without loss of generality, we suggest that the solutions $P_1(x,t)$ and $P_2(x,t)$ of the coupled differential equations (19a) and (19b) can be put in more explicit forms as follows:

$$P_1(x,t) = T^T BM(x,t)B^T T + \begin{bmatrix} K_1 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix} \tag{35a}$$

$$P_2(x,t) = T^T BM(x,t)B^T T + \begin{bmatrix} K_2 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix} \tag{35b}$$

where K_1 and K_2 are some positive-definite symmetric constant matrices and $B = [I_{3\times 3} \;\; 0_{3\times 3}]^T$. In the following paragraphs, it is demonstrated that, under some conditions, these suggested matrices $P_1(x,t)$ and $P_2(x,t)$ are the solutions of differential equations (19a) and (19b). Furthermore, the constant matrices T, K_1, and K_2 can be solved from a pair of coupled algebraic equations.

Consider the second and third terms on the left-hand side of differential equations (19a) and (19b). Using the skew-symmetric property in Eq. (6) and the selected relations in Eqs. (35a) and (35b), we get

$$\dot{P}_1(x,t) + P_1(x,t)A_T(x,t) + A_T^T(x,t)P_1(x,t) = \begin{bmatrix} 0_{3\times 3} & K_1 \\ K_1 & 0_{3\times 3} \end{bmatrix} \tag{36a}$$

$$\dot{P}_2(x,t) + P_2(x,t)A_T(x,t) + A_T^T(x,t)P_2(x,t) = \begin{bmatrix} 0_{3\times 3} & K_2 \\ K_2 & 0_{3\times 3} \end{bmatrix} \tag{36b}$$

It can also be easily checked that

$$B_T^T(x,t)P_1(x,t) = \lambda_2 B^T T \tag{37a}$$

$$B_T^T(x,t)P_2(x,t) = \lambda_2 B^T T \tag{37b}$$

which satisfy the condition in Eq. (19c).

By the use of the results of Eqs. (36a–37b), coupled differential equations (19a) and (19b) can be reduced to the following coupled algebraic equations:

$$\begin{bmatrix} 0_{3\times 3} & K_1 \\ K_1 & 0_{3\times 3} \end{bmatrix} + Q_1 - \lambda_2^2 T^T B\left(W^{-1} - \frac{1}{\gamma^2}I_{3\times 3}\right)B^T T = 0 \tag{38a}$$

$$\begin{bmatrix} 0_{3\times 3} & K_2 \\ K_2 & 0_{3\times 3} \end{bmatrix} + Q_2 - \lambda_2^2 T^T B\left(W^{-1} - \frac{2}{\gamma^2}I_{3\times 3}\right)B^T T = 0 \tag{38b}$$

In addition, the optimal control law, adaptive law, and the worst-case disturbance can be rewritten as

$$u_e^*(x,t) = -\lambda_2 W^{-1}r(t) \tag{39a}$$

$$\dot{\hat{\xi}}(x,t) = \lambda_2 S^{-1}\Phi^T(x,t)r(t) \tag{39b}$$

$$d^*(x,t) = \left(\lambda_2/\gamma^2\right)r(t) \tag{39c}$$

where $r(t)$ is in Eq. (9a). It is obvious that Eqs. (38a–39c) are all based on known matrices or variables and are applicable. From the preceding analysis, matrices $P_1(x,t)$ and $P_2(x,t)$ in Eqs. (35a) and (35b) are the solutions of coupled differential equations (19a) and (19b) if matrices K_1, K_2, and T satisfy coupled algebraic equations (38a) and (38b) simultaneously. Furthermore, the positive-definite symmetric property of K_1 and K_2 must be satisfied. To guarantee the solvability, further assumptions and constraints on the weighting matrices Q_1, Q_2, and W are required.

For the simplicity of design, let

$$W = a^2 I_{3\times 3}, \qquad S = bI_{6\times 6}, \qquad Q_2 = \alpha Q_1 \tag{40}$$

where $a > 0$, $b > 0$, and $\alpha > 0$ and the positive-definite symmetric matrix Q_1 can be factorized by the Cholesky factorization (see Ref. 21) as

$$Q_1 = \begin{bmatrix} q_{11}^2 I_{3\times 3} & Q_{12} \\ Q_{12}^T & q_{22}^2 I_{3\times 3} \end{bmatrix} \tag{41}$$

With the definitions of T and B in Eqs. (9b) and (11) and the forms in Eqs. (40) and (41), the coupled algebraic equations in Eqs. (38a) and (38b) can be solved by the following equalities:

$$\lambda_1 = \frac{\sqrt{a\gamma}q_{11}}{4\sqrt{q_{22}^2(\gamma^2 - a^2)}} \tag{42a}$$

$$\lambda_2 = \frac{q_{22}}{q_{11}}\lambda_1 \tag{42b}$$

$$\alpha = \frac{\gamma^2 - 2a^2}{\gamma^2 - a^2} \tag{42c}$$

$$K_1 = q_{11}q_{22}I_{3\times 3} - Q_{12} \tag{42d}$$

$$K_2 = \alpha K_1 \tag{42e}$$

with $0 < a < \gamma/\sqrt{2}$ and $Q_{12} < q_{11}q_{22}I_{3\times 3}$.

From this analysis, the solution to the adaptive mixed H_2/H_∞ attitude control problem is concluded in the following corollary.

Corollary: For the adaptive mixed H_2/H_∞ attitude control, given a desired disturbance attenuation level $\gamma > 0$, let the weighting matrix W, S, Q_1, and Q_2 be taken as in Eqs. (40) and (41) with α satisfying the requirement in Eq. (42c) and $Q_{12} < q_{11}q_{22}I_{3\times 3}$. If the constant a in the weighting matrix W satisfies

$$0 < a < \gamma\big/\sqrt{2} \tag{43}$$

then the following adaptive mixed H_2/H_∞ control law solves the adaptive mixed H_2/H_∞ attitude control problem

$$\boldsymbol{u}^*(\boldsymbol{x}, t) = \boldsymbol{u}_e^*(\boldsymbol{x}, t) - \Phi(\boldsymbol{x}, t)\hat{\boldsymbol{\xi}}(\boldsymbol{x}, t) \tag{44a}$$

where

$$\boldsymbol{u}_e^*(\boldsymbol{x}, t) = -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}(q_{11}\theta + q_{22}\dot{\theta}) \tag{44b}$$

$$\dot{\hat{\boldsymbol{\xi}}}(\boldsymbol{x}, t) = \frac{a\gamma}{b\sqrt{\gamma^2 - a^2}}\Phi^T(\boldsymbol{x}, t)(q_{11}\theta + q_{22}\dot{\theta}) \tag{44c}$$

Also, the worst-case disturbance is

$$d^*(\boldsymbol{x}, t) = \left(a/\gamma\sqrt{\gamma^2 - a^2}\right)(q_{11}\theta + q_{22}\dot{\theta}) \tag{44d}$$

□

Remark 5: The designed optimal control law in Eq. (44b) can be viewed as a PD control law

$$\boldsymbol{u}_e^*(\boldsymbol{x}, t) = K_p\theta + K_d\dot{\theta} \tag{45}$$

with the control gains $K_p = -\gamma q_{11}/[a\sqrt{(\gamma^2 - a^2)}]$ and $K_d = -\gamma q_{22}/[a\sqrt{(\gamma^2 - a^2)}]$, both of which are adjusted by the attenuation level γ according to the mission desired. □

Design Algorithm

Based on the preceding discussion, the proposed spacecraft adaptive attitude control design can be outlined as the following design algorithm.

The adaptive mixed H_2/H_∞ attitude control design algorithm consists of three steps:

1) Choose a desired level of disturbance attenuation, $\gamma > 0$.

2) Select the weighting matrices, $W = a^2 I_{3\times 3}$ and $S = bI_{6\times 6}$, such that $0 < a < \gamma/\sqrt{2}$ and $b > 0$

$$Q_1 = \begin{bmatrix} q_{11}^2 I_{3\times 3} & Q_{12} \\ Q_{12}^T & q_{22}^2 I_{3\times 3} \end{bmatrix}$$

with $Q_{12} < q_{11}q_{22}I_{3\times 3}$ and $Q_2 = [(\gamma^2 - 2a^2)/(\gamma^2 - a^2)]Q_1$.

3) Obtain the corresponding adaptive mixed H_2/H_∞ applied torque associated with the update law for the spacecraft system in Eqs. (1) and (2)

$$\tau_a = R^{-T}(\theta)\left[-\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}(q_{11}\theta + q_{22}\dot{\theta}) - \Phi(\boldsymbol{x}, t)\hat{\boldsymbol{\xi}}\right] \tag{46a}$$

$$\dot{\hat{\boldsymbol{\xi}}} = \frac{a\gamma}{b\sqrt{\gamma^2 - a^2}}\Phi^T(\boldsymbol{x}, t)(q_{11}\theta + q_{22}\dot{\theta}) \tag{46b}$$

where the regression matrix $\Phi(\boldsymbol{x}, t)$ is defined in the Appendix. □

Remark 6: By following Remark 4 and using the same techniques as in the preceding paragraphs, we have the following results.

1) For the adaptive H_2 attitude control design case, the applied torque and the update law for the spacecraft system in Eqs. (1) and (2) are

$$\tau_a = R^{-T}(\theta)[-(1/a)(q_{11}\theta + q_{22}\dot{\theta}) - \Phi(\boldsymbol{x}, t)\hat{\boldsymbol{\xi}}] \tag{47a}$$

$$\dot{\hat{\boldsymbol{\xi}}} = (a/b)\Phi^T(\boldsymbol{x}, t)(q_{11}\theta + q_{22}\dot{\theta}) \tag{47b}$$

with $a > 0$ and $b > 0$.

2) For the adaptive H_∞ attitude control design case, the applied torque and the update law for the spacecraft system in Eqs. (1) and (2) are

$$\tau_a = R^{-T}(\theta)\left[-\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}(q_{11}\theta + q_{22}\dot{\theta}) - \Phi(\boldsymbol{x}, t)\hat{\boldsymbol{\xi}}\right] \tag{48a}$$

$$\dot{\hat{\boldsymbol{\xi}}} = \frac{a\gamma}{b\sqrt{\gamma^2 - a^2}}\Phi^T(\boldsymbol{x}, t)(q_{11}\theta + q_{22}\dot{\theta}) \tag{48b}$$

with $0 < a < \gamma$ and $a > 0$. □

Simulation Results

The experimental simulations on the Republic of China Satellite-1 (ROCSAT-1) spacecraft (Fig. 1) have been made with the assistance of the National Space Program Office in Taiwan, Republic of China. To substantiate the performance of the adaptive mixed H_2/H_∞ attitude controller design, we consider the unknown inertia matrix of the form

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{bmatrix}$$

and the unknown parameter vector to be estimated is $\boldsymbol{\xi} = [J_{11}\ J_{12}\ J_{13}\ J_{22}\ J_{23}\ J_{33}]^T$ with the initial value $\hat{\boldsymbol{\xi}}(0) = [126.98\ -1.87\ 3.38\ 116.63\ -2.40\ 209.36]^T$. By Eq. (7), we can calculate the regression matrix $\Phi[\theta, \dot{\theta}, (\lambda_1/\lambda_2)\theta, (\lambda_1/\lambda_2)\dot{\theta}]$ of dimension 3×6 (see the Appendix). The orbital rate for the ROCSAT-1 spacecraft at its nominal 600-km circular orbit environment is $\omega_0 = 0.0011$ rad/s. The external disturbances $\boldsymbol{\tau}_{\text{aero}}$ and $\boldsymbol{\tau}_{\text{solar}}$ in the body frame are shown in Fig. 2. We consider that the attitude state $x = [\theta\ \dot{\theta}]^T$ of spacecraft, with initial values at $\theta(0) = (\pi/45, \pi/45, \pi/45)^T$ and $\dot{\theta}(0) = (0, 0, 0)^T$, is required to decay to zero. Moreover, for the ROCSAT-1 spacecraft, the output torque vector of the reaction wheel is limited in amplitude due to saturation. Thus, we set the saturation values $S_{ai} = 0.015$ N · m, $i = 1, 2, 3$ in Eq. (3).

Simulation 1

To verify the ability for disturbance attenuation of the proposed method, three cases of control designs are considered: adptive H_2 attitude control design, adaptive H_∞ attitude control design, and the adaptive mixed H_2/H_∞ attitude control design. In all cases, we

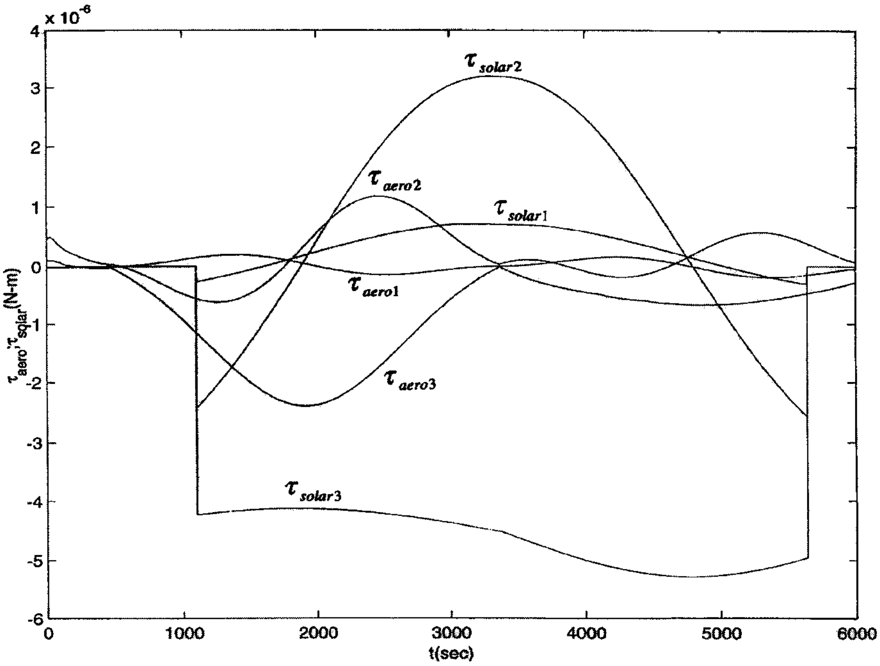

Fig. 2 External disturbances $\tau_{\rm aero}$ and $\tau_{\rm solar}$ of the ROCSAT-1 spacecraft at its nominal circular orbit environment.

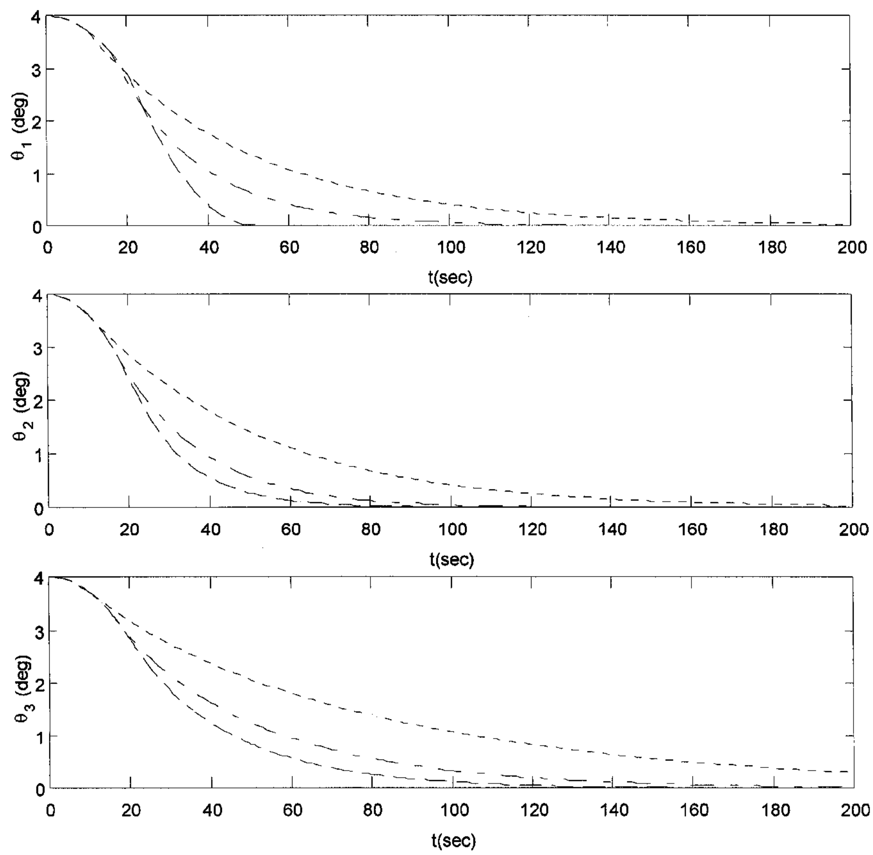

Fig. 3 Attitude angles: ——, for adaptive H_2 (a = 0.6); –·–, for adaptive H_∞ (γ = 0.6, a = 0.42); and - - - -, for adaptive mixed H_2/H_∞ (γ = 0.6, α = 0.5, a = 0.3464).

choose the weighting matrix S to be the identity matrix $I_{6\times 6}$, that is, $b = 1$. Other simulation parameters are selected as follows:

1) For case 1, adaptive H_2 optimal attitude control, select $a = 0.6$ and $Q = I_{6\times 6}$.

2) For case 2, adaptive H_∞ attitude control, select $\gamma = 0.6$, $a = 0.42$, and $Q = I_{6\times 6}$.

3) For case 3, adaptive mixed H_2/H_∞ attitude control, select $\gamma = 0.6$, $Q_1 = I_{6\times 6}$, and $a = 0.3464$. Then, from Eq. (42c), we select $\alpha = 0.5$.

For the adaptive H_2 optimal attitude control case, the applied torque and update law in Eqs. (47a) and (47b) are used. For the adaptive H_∞ attitude control case, the applied torque and update law in Eqs. (48a) and (48b) are used. For the adaptive mixed H_2/H_∞ attitude control case, the applied torque and update law in Eqs. (46a) and (46b) are used. The simulation results are shown in Figs. 3–7. The attitude angles θ_1, θ_2, and θ_3 are represented in Fig. 3, and the attitude angle rates $\dot\theta_1$, $\dot\theta_2$, and $\dot\theta_3$ are depicted in Fig. 4 for the preceding three cases. As the results of these three proposed methods

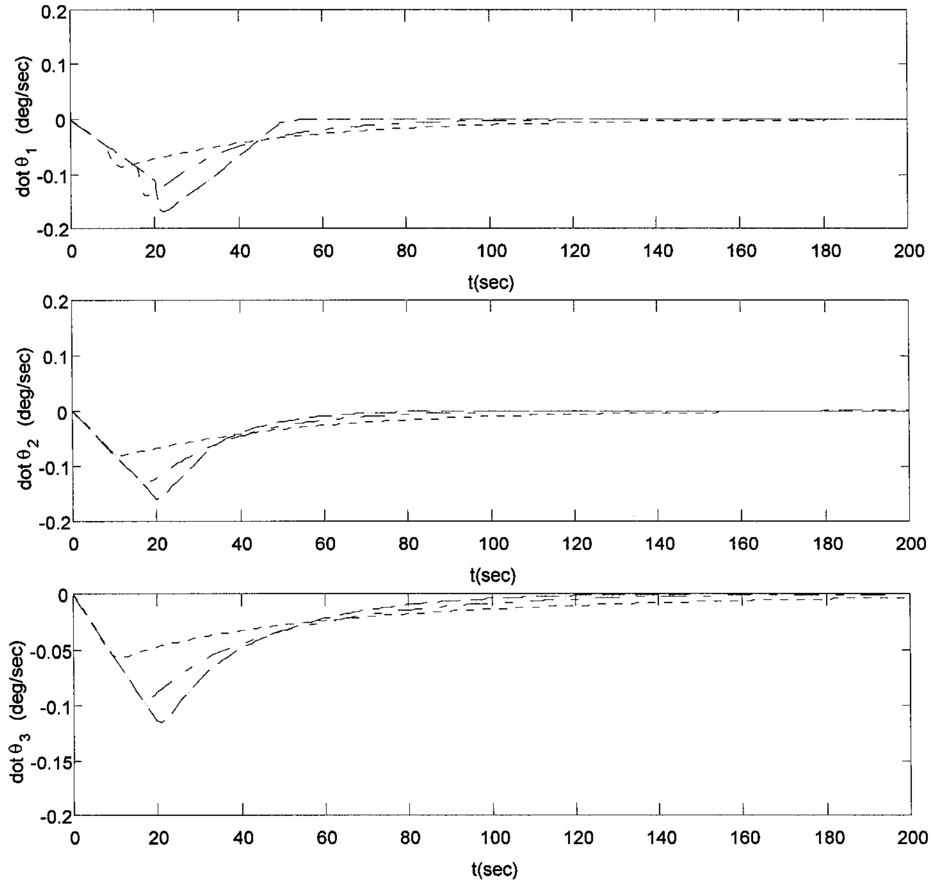

Fig. 4 Attitude angles rates: ——, for adaptive H_2 (a = 0.6); –·–, for adaptive H_∞ ($\gamma = 0.6, a = 0.42$); and - - - -, for adaptive mixed H_2/H_∞ ($\gamma = 0.6$, $\alpha = 0.5, a = 0.3464$).

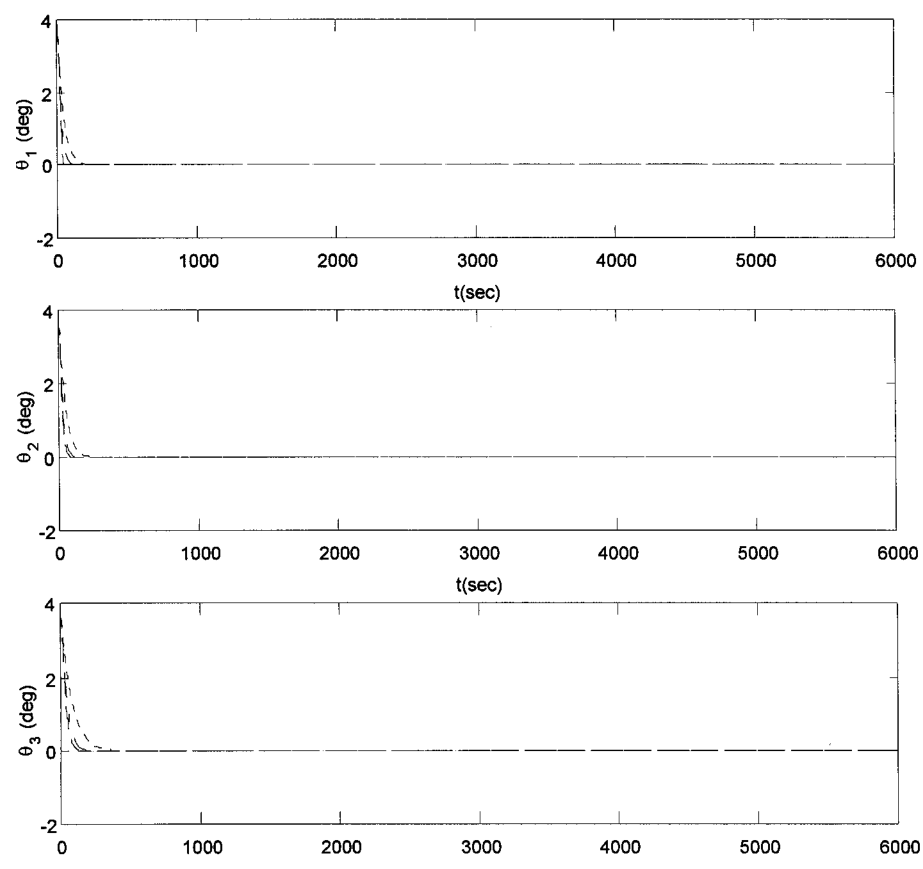

Fig. 5 Steady-state response of attitude angles: ——, for adaptive H_2 (a = 0.6); –·–, for adaptive H_∞ ($\gamma = 0.6, a = 0.42$); and - - - -, for adaptive mixed H_2/H_∞ ($\gamma = 0.6, \alpha = 0.5, a = 0.3464$).

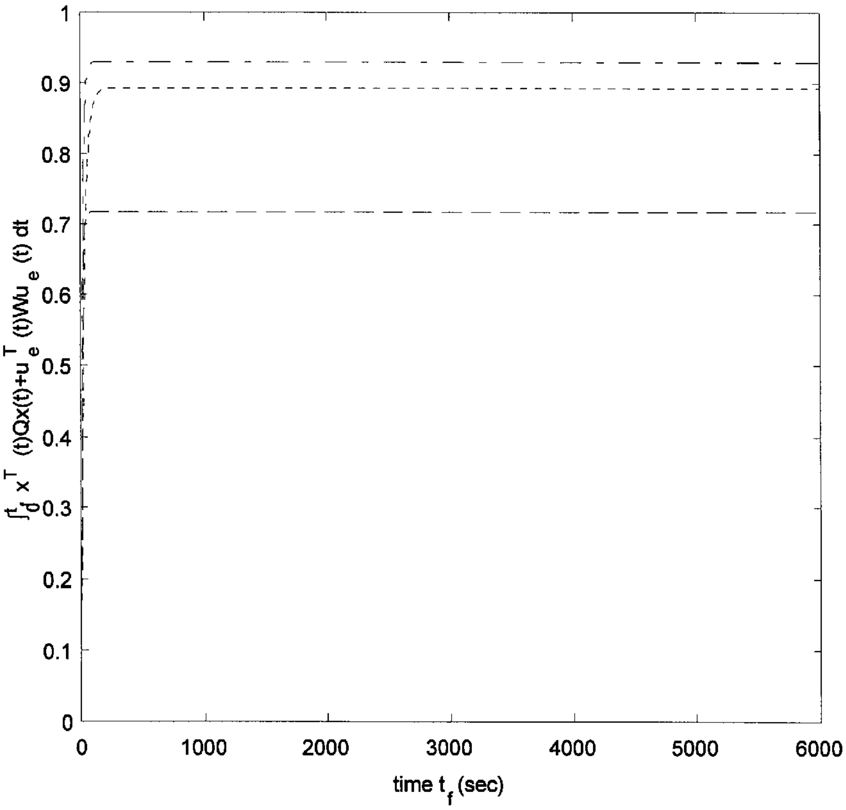

Fig. 6 Attitude tracking performances: ——, for adaptive H_2 ($a = 0.6$); –·–, for adaptive H_∞ ($\gamma = 0.6, a = 0.42$); and - - - -, for adaptive mixed H_2/H_∞ ($\gamma = 0.6, \alpha = 0.5, a = 0.3464$).

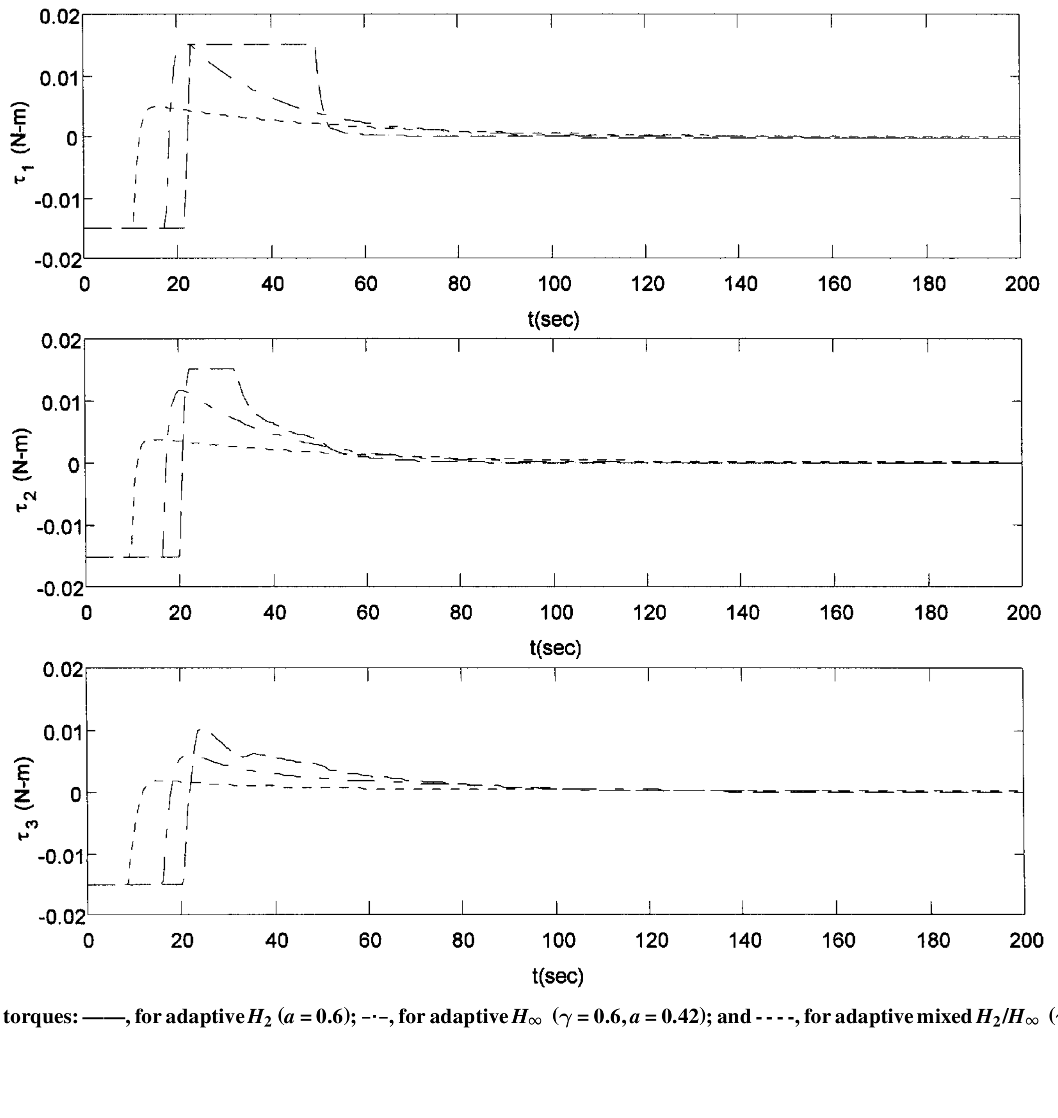

Fig. 7 Applied torques: ——, for adaptive H_2 ($a = 0.6$); –·–, for adaptive H_∞ ($\gamma = 0.6, a = 0.42$); and - - - -, for adaptive mixed H_2/H_∞ ($\gamma = 0.6, \alpha = 0.5, a = 0.3464$).

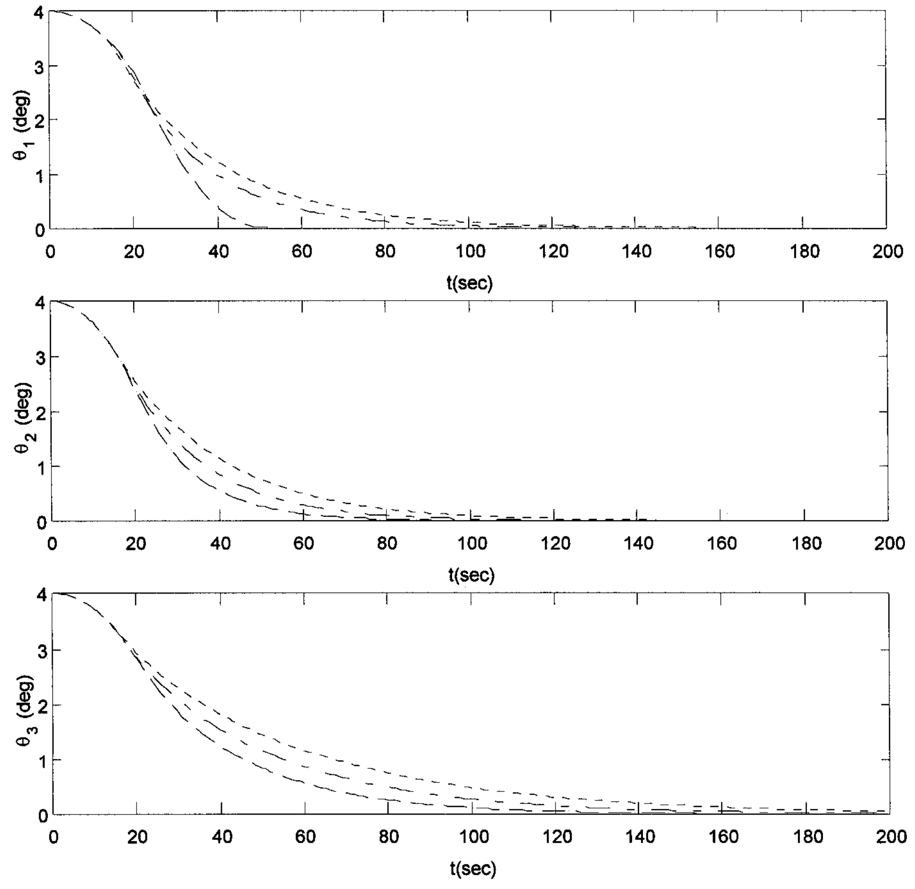

Fig. 8 Attitude angles: ——, for adaptive mixed H_2/H_∞ ($\gamma = 0.8, \alpha = 0.5$); –·–, for adaptive mixed H_2/H_∞ ($\gamma = 0.7, \alpha = 0.5$); and - - - -, for adaptive mixed H_2/H_∞ ($\gamma = 0.6, \alpha = 0.5$).

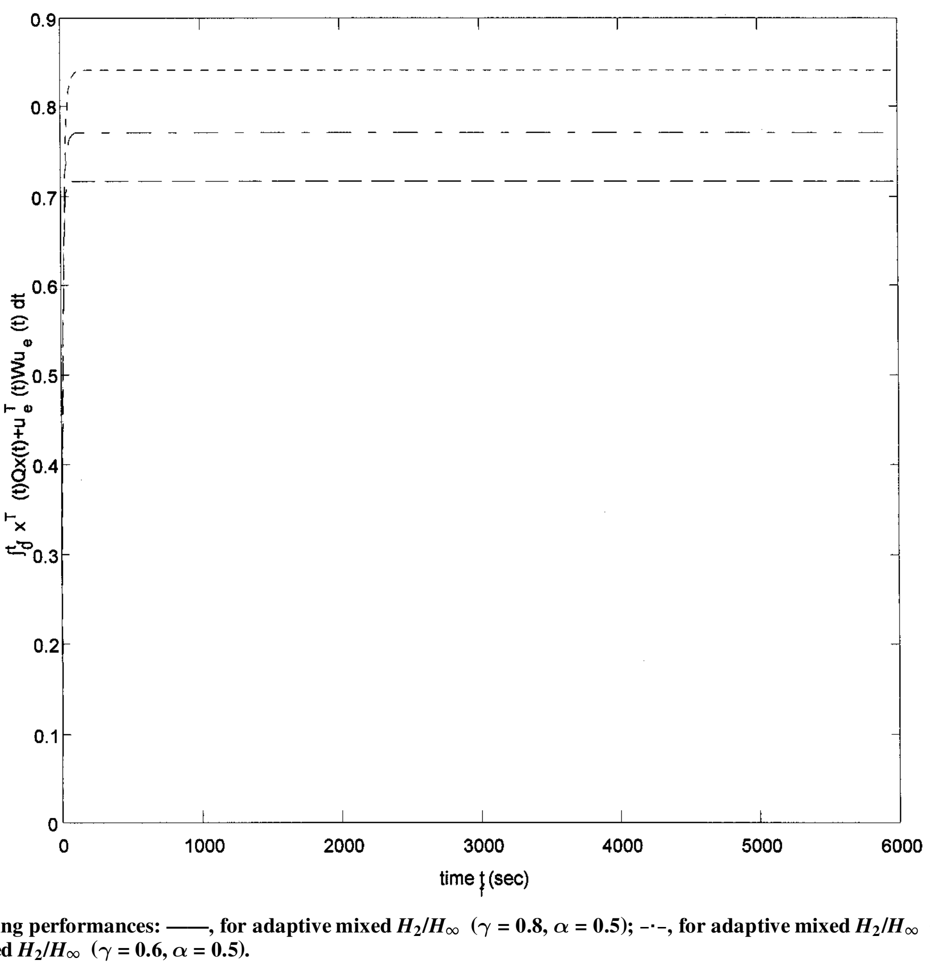

Fig. 9 Attitude tracking performances: ——, for adaptive mixed H_2/H_∞ ($\gamma = 0.8, \alpha = 0.5$); –·–, for adaptive mixed H_2/H_∞ ($\gamma = 0.7, \alpha = 0.5$); and - - - -, for adaptive mixed H_2/H_∞ ($\gamma = 0.6, \alpha = 0.5$).

reveal, the adaptive mixed H_2/H_∞ attitude control causes quicker decay responses and has superior ability to diminish the effect of an external disturbance under unknown parameters. These results can be expected because the adaptive mixed H_2/H_∞ attitude controller is designed to achieve the adaptive H_2 attitude control under an adaptive H_∞ disturbance attenuation constraint. Furthermore, the adaptive H_2 attitude controller causes the fewest disturbance attenuation abilities (with respect to slower decay responses) among the three methods. This is reasonable because the adaptive H_2 attitude control is designed without consideration of the external disturbance and, therefore, does not, in general, guarantee any robust performance in the face of disturbance. The simulation result in Fig. 5 shows the steady-state response of attitude angles θ_1, θ_2, and θ_3 over $t = 6000$ s. It can be found that the effects of external disturbances to the attitude state are attenuated, even when the external disturbances are at a maximum at $t = 3000$ s. The attitude control performance

$$\int_0^{t_f} \boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W\boldsymbol{u}_e(t)\,\mathrm{d}t$$

is plotted in Fig. 5 for the three cases. The applied torques $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\boldsymbol{\tau}_3$ are represented in Fig. 7. It can be seen that the controller that produces better performance (as the adaptive mixed H_2/H_∞ attitude controller) takes larger control torque in amplitude. This is a tradeoff problem between the attitude control performance and the amplitude of control torque. These simulation results show that the performance of the adaptive H_∞ attitude controller is better than the performance of adaptive H_2 attitude control and worse than the performance of adaptive mixed H_2/H_∞ attitude control.

Simulation 2

To illustrate the robust capability of disturbance attenuation of the proposed adaptive mixed H_2/H_∞ attitude control design, the adaptive mixed H_2/H_∞ attitude control torque and update law in Eqs. (46a) and (46b) are designed to possess the following three different disturbance attenuation levels:

1) For case 1, select $\gamma = 0.8$, $Q_1 = I_{6\times 6}$, and $a = 0.462$. Then $\alpha = 0.5$.

2) For case 2, select $\gamma = 0.7$, $Q_1 = I_{6\times 6}$, and $a = 0.404$. Then $\alpha = 0.5$.

3) For case 3, select $\gamma = 0.6$, $Q_1 = I_{6\times 6}$, and $a = 0.346$. Then $\alpha = 0.5$.

The simulation results are shown in Figs. 8 and 9 for the attitude angles and the performance

$$\int_0^{t_f} \boldsymbol{x}^T(t)Q_1\boldsymbol{x}(t) + \boldsymbol{u}_e^T(t)W\boldsymbol{u}_e(t)\,\mathrm{d}t$$

respectively. It is obvious that a smaller γ may yield a better performance [lower cost function $J_1(\boldsymbol{u}_e, d)$] in attenuating the effect of external disturbances $d(t)$.

Conclusions

This paper presents an adaptive mixed H_2/H_∞ attitude control design of the nonlinear spacecraft system with an external disturbance and uncertain inertia matrix. In practical situations, the inertia matrix is uncertain and external disturbance is inevitable. By the adaptive control method, the uncertain parameters are estimated. Then, by the mixed H_2 and H_∞ controller design, the effect of an external disturbance to the spacecraft attitude can be restrained and the attitude state, as well as consumed energy of the controller, is minimized. Unlike the conventional nonlinear mixed H_2/H_∞ control design, which is based on solving two coupled differential equations, a general solution can be obtained by the method proposed in this paper via skew-symmetric property and state transformation techniques. The structure of the controller is very simple and can be viewed as an adaptive PD controller with the controller gains depending on the disturbance attenuation level γ, which is assigned according to mission requirement. According to the simulation results, the adaptive mixed H_2/H_∞ attitude controller has greater ability to diminish the effects of an external disturbance to achieve better performance than the adaptive H_2 attitude controller or the adaptive H_∞ attitude controller. Moreover, from the control laws in Eqs. (46a) and (48a), we could see that both equations are the same with the exception that they differ in the constraints on γ, that is, H_∞ control with $\gamma > a$ but mixed H_2/H_∞ control with $\gamma > \sqrt{(2)}a$. We can conclude that the solutions of the mixed H_2/H_∞ control are contained in the solutions of the H_∞ control. Furthermore, the adaptive mixed H_2/H_∞ attitude controller achieves robust performance for the spacecraft attitude control systems. From the experimental simulation results on the ROCSAT-1 spacecraft system, the proposed design algorithm exhibits significant advantages for the attitude control of the spacecraft system under unknown parameters and a large external disturbance.

Appendix: Determining the Regression Matrix

By Eq. (7), the regression matrix $\Phi(\boldsymbol{x}, t) = [\phi_{ij}]$ for $i = 1, 2, 3$ and $j = 1, 2, \ldots, 6$ can be determined as

$$\phi_{11} = \dot{y}_1 - s_2\dot{y}_3 - c_2\dot{\theta}_2 y_3 + B_1$$

$$\phi_{12} = c_1\dot{y}_2 + s_1c_2\dot{y}_3 + s_1(A_1 - \dot{\theta}_1)y_2 + (\dot{\theta}_1c_1c_2 - \dot{\theta}_2s_1s_2 - c_1c_2A_1)y_3 + B_2 - \omega_0(s_1c_3 - c_1s_2s_3)A_1$$

$$\phi_{13} = -s_1\dot{y}_2 + c_1c_2\dot{y}_3 - c_1(\dot{\theta}_1 - A_1)y_2 - [s_1c_2(\dot{\theta}_1 - A_1) + \dot{\theta}_2c_1s_2]y_3 + B_3 - \omega_0(c_1c_3 + s_1s_2s_3)A_1$$

$$\phi_{14} = s_1A_2y_2 - c_1c_2A_2y_3 - \omega_0(s_1c_3 - c_1s_2s_3)A_2$$

$$\phi_{15} = (c_1A_2 + s_1A_3)y_2 - (c_1A_3 - s_1A_2)c_2y_3 - \omega_0[(c_1c_3 + s_1s_2s_3)A_2 + (s_1c_3 - c_1s_2s_3)A_3]$$

$$\phi_{16} = c_1A_3y_2 + s_1c_2A_3y_3 - \omega_0(c_1c_3 + s_1s_2s_3)A_3$$

$$\phi_{21} = c_2A_1y_3 - \omega_0s_2s_3A_1$$

$$\phi_{22} = c_1\dot{y}_1 - c_1s_2\dot{y}_3 - s_1A_1y_1 - (c_1c_2\dot{\theta}_2 - c_2A_2 - s_1s_2A_1)y_3 + B_1c_1 + \omega_0[c_2s_3s_1A_1 - s_2s_3A_2]$$

$$\phi_{23} = -s_1\dot{y}_1 + s_1s_2\dot{y}_3 - c_1A_1y_1 + (s_1c_2\dot{\theta}_2 + c_2A_3 + c_1s_2A_1)y_3 - B_1s_1 + \omega_0(c_1c_2s_3A_1 - s_2s_3A_3)$$

$$\phi_{24} = c_1^2\dot{y}_2 + c_1s_1c_2\dot{y}_3 - s_1A_2y_1 - s_1c_1\dot{\theta}_1y_2 + [c_1(c_1c_2\dot{\theta}_1 - s_1s_2\dot{\theta}_2) + s_1s_2A_2]y_3 + B_2c_1 + \omega_0s_1c_2s_3A_2$$

$$\phi_{25} = -2c_1s_1\dot{y}_2 + c_2\left(c_1^2 - s_1^2\right)\dot{y}_3 - (c_1A_2 + s_1A_3)y_1 + \left(s_1^2 - c_1^2\right)\dot{\theta}_1y_2 + \left[-2c_1s_1c_2\dot{\theta}_1 + s_2\left(s_1^2 - c_1^2\right)\dot{\theta}_2 + (s_1s_2A_3 + c_1s_2A_2)\right]y_3 - B_2s_1 + B_3c_1 + \omega_0c_2s_3(c_1A_2 + s_1A_3)$$

$$\phi_{26} = s_1^2\dot{y}_2 - s_1c_1c_2\dot{y}_3 - c_1A_3y_1 + s_1c_1\dot{\theta}_1y_2 + [s_1(s_1c_2\dot{\theta}_1 + c_1s_2\dot{\theta}_2) + c_1s_2A_3]y_3 - B_3s_1 + \omega_0c_1c_2s_3A_3$$

$$\phi_{31} = -s_2\dot{y}_1 + s_2^2\dot{y}_3 - c_2A_1y_2 + c_2s_2\dot{\theta}_2y_3 - B_1s_2 + \omega_0c_2c_3A_1$$

$$\phi_{32} = s_1c_2\dot{y}_1 - s_2c_1\dot{y}_2 - 2s_1s_2c_2\dot{y}_3 + c_1c_2A_1y_1 + (s_1s_2\dot{\theta}_1 - s_1s_2A_1 - c_2A_2)y_2 + \left[-c_1c_2s_2\dot{\theta}_1 + s_1\left(s_2^2 - c_2^2\right)\dot{\theta}_2\right]y_3 + B_1s_1c_2 - B_2s_2 + \omega_0[c_2c_3A_2 + (s_1s_2c_3 - c_1s_3)A_1]$$

$$\phi_{33} = c_1c_2\dot{y}_1 + s_1s_2\dot{y}_2 - 2c_1c_2s_2\dot{y}_3 - s_1c_2A_1y_1 + (c_1s_2\dot{\theta}_1 - c_1s_2A_1 - c_2A_3)y_2 + \left[s_1c_2s_2\dot{\theta}_1 + c_1\left(s_2^2 - c_2^2\right)\dot{\theta}_2\right]y_3 + B_1c_1c_2 - B_3s_2 + \omega_0[c_2c_3A_3 + (c_1s_2c_3 + s_1s_3)A_1]$$

$$\phi_{34} = s_1c_1c_2\dot{y}_2 + s_1^2c_2^2\dot{y}_3 + c_1c_2A_2y_1 - \left(s_1s_2A_2 + s_1^2c_2\dot{\theta}_1\right)y_2$$

$$+ s_1c_2(c_1c_2\dot\theta_1 - s_1s_2\dot\theta_2)y_3 + B_2s_1c_2 + \omega_0(s_1s_2c_3 - c_1s_3)A_2$$

$$\phi_{35} = c_2\left(c_1^2 - s_1^2\right)\dot y_2 + 2s_1c_1c_2^2\dot y_3 - c_2(s_1A_2 - c_1A_3)y_1 - [2c_1s_1c_2\dot\theta_1 + s_2(s_1A_3 + c_1A_2)]y_2 + \left[c_2^2\left(c_1^2 - s_1^2\right)\dot\theta_1 - 2c_1s_1c_2s_2\dot\theta_2\right]y_3 + B_2c_1c_2 + B_3s_1c_2 + \omega_0[(s_1s_2c_3 - c_1s_3)A_3 + (c_1s_2c_3 + s_1s_3)A_2]$$

$$\phi_{36} = -c_1s_1c_2\dot y_2 + c_1^2c_2^2\dot y_3 - s_1c_2A_3y_1 - \left(c_1^2c_2\dot\theta_1 + c_1s_2A_3\right)y_2 - c_1c_2(s_1c_2\dot\theta_1 + c_1s_2\dot\theta_2)y_3 + B_3c_1c_2 + \omega_0(c_1s_2c_3 + s_1s_3)A_3$$

Here, we use the short-hand notations

$$c_1 = \cos\theta_1, \qquad c_2 = \cos\theta_2, \qquad c_3 = \cos\theta_3, \qquad s_1 = \sin\theta_1$$

$$s_2 = \sin\theta_2, \qquad s_3 = \sin\theta_3, \qquad A_1 = \dot\theta_1 - s_2\dot\theta_3 - \omega_0c_2s_3$$

$$A_2 = c_1\dot\theta_2 + s_1c_2\dot\theta_3 - \omega_0(c_1c_3 + s_1s_2s_3)$$

$$A_3 = -s_1\dot\theta_2 + c_1c_2\dot\theta_3 - \omega_0(c_1s_2s_3 - s_1c_3)$$

$$B_1 = \omega_0(s_2s_3\dot\theta_2 - c_2c_3\dot\theta_3)$$

$$B_2 = \omega_0[(s_1c_3 - c_1s_2s_3)\dot\theta_1 - s_1c_2s_3\dot\theta_2 + (c_1s_3 - s_1s_2c_3)\dot\theta_3]$$

$$B_3 = \omega_0[(c_1c_3 + s_1s_2s_3)\dot\theta_1 - c_1c_2s_3\dot\theta_2 - (s_1s_3 + c_1s_2c_3)\dot\theta_3]$$

$$y = [y_1 \quad y_2 \quad y_3]^T = (\lambda_1/\lambda_2)\theta$$

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